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Asymptotic Properties of Sequences of Iterates of Nonlinear Transformations

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By considering functions defined on the unit interval with a single zero minimum and a single unit maximum we are led to a version of the doubling or universal transformation. The fixed point functions of this doubling transformation have certain invariance properties under conjugacy. These invariance properties lead to a widening of the concept of universality to power law conjugacy classes in which the Feigenbaum divergence parameter δ is a function only of the product of the powers with which iterating functions approach unity at the maximum and zero at the minimum. We also construct an effective method for computing the divergence parameter from iterates, and derivatives of iterates, generated by the appropriate fixed point function.

KEY WORDS: Maps on an interval; iteration; conjugacy; universal; fixed point function; functional equation; doubling transformation.

1. INTRODUCTION

The behavior of first-order difference equations

$$x_{n+1} = \phi(x_n) \tag{1.1}$$

where $\phi(x)$ has the form shown in Fig. 1 has been the subject of much recent study.[1] Even for simple cases, such as

$$\phi(x) = \lambda x e^{-x} \quad \text{and} \quad \phi(x) = \lambda x(1-x) \tag{1.2}$$

which have appeared extensively in the biological literature, the behavior can be quite intricate and exotic.[2]

Typically, when ϕ depends on a parameter λ that controls the slope of

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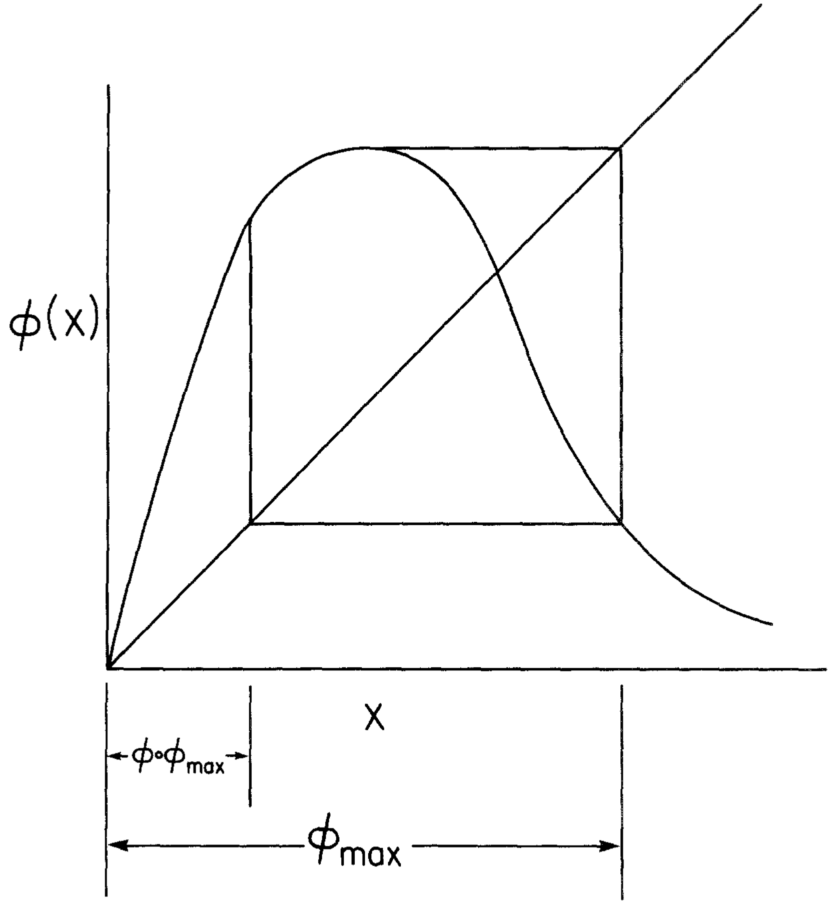

Fig. 1. Relevant part of the function on the interval $\phi(\phi_{\max}) \leqslant x \leqslant \phi_{\max}$.

ϕ at its (single) nontrivial fixed point x^* [as in (1.2), for example], one finds that x^* is stable (that is, the iterates x_n tend to x^* as n tends to infinity) for a range of values of λ until $\phi'(x^*)$ passes through -1, at which point, corresponding to $\lambda = \lambda_1$ say, the fixed point bifurcates into two fixed points x, y of period 2, or equivalently to a two-cycle satisfying

$$x = \phi(y) \quad \text{and} \quad y = \phi(x) \tag{1.3}$$

or in terms of the second iterate $\phi^{(2)}$ of the function ϕ defined by functional composition as

$$\phi^{(2)}(x) \equiv \phi \circ \phi(x) \equiv \phi(\phi(x)) \tag{1.4}$$

$$x = \phi^{(2)}(x) \quad \text{and} \quad y = \phi^{(2)}(y) \tag{1.5}$$

Thereafter one gets a cascade of bifurcations to 2^n-cycles occurring at parameter values λ_n, $n = 1, 2, \ldots$ that approach a limiting value $\lambda_\infty < \infty$ as n tends to infinity. Beyond λ_∞, other even cycles appear followed by odd and even cycles. In some cases cycles of all orders are present while in other cases no cycles of any order are present. Such behavior is usually referred to as "chaotic."[2]

An interesting universal feature of the sequence $\{\lambda_n\}$ was discovered recently by Feigenbaum.[3] He found that

$$|\lambda_n - \lambda_\infty| \sim \delta^{-n} \qquad \text{as } n \to \infty \tag{1.6}$$

where δ apparently depends only on the shape of $\phi(x)$ near its (single) maximum. For example, in the usual case of a quadratic maximum [such as Eqs. (1.2)] Feigenbaum found numerically that $\delta = 4.6692016\ldots$. He also discovered that suitably scaled 2^n-iterates of $\phi(x)$ approach a universal function with a universal scale factor of $\alpha = 2.5029079\ldots$ for the quadratic maximum examples. Recently, some of these ideas have been made rigorous by Collet, Eckmann, and Lanford[4] when $\phi(x)$ near its maximum (at x_m) has the shape $|x - x_m|^{1+\epsilon}$ for ϵ sufficiently small. Lanford[5] has also reported progress in the more usual case of $\epsilon = 1$.

Our purpose here is to discuss some features of this apparent universal behavior by focusing attention on those properties of sequences of iterates $\{x_n\}$ obtained from (1.1) that are preserved when such sequences become arbitrarily long. In this respect it is clear that one can immediately restrict one's attention to conjugacy classes of functions consisting of functions $\hat{\phi}$ and ϕ related by a conjugacy defined by

$$\hat{\phi} = g \circ \phi \circ g^{-1} \tag{1.7}$$

for some invertible g. Thus it is clear from (1.7) that properties of iterates of one function (ϕ or $\hat{\phi}$) can be obtained immediately from the other.

In the following section we present a variant of the so-called "doubling transformation" (T) that is designed to easily incorporate conjugacies, and in particular power law conjugacies [$g(x) = x^p$ in (1.7)]. Power law conjugacies are shown in Section 3 to be the only relevant conjugacies so far as asymptotic properties of sequences of iterates are concerned.

The parameter δ [Eq. (1.6)] governing the local divergence from a fixed point of T is shown in Section 4 to be invariant under a power law conjugacy. This leads immediately to a modified view of universality in which the form of the function near its maximum is not the only feature determining the value of the divergence parameter δ.

In Section 5 we present an alternative approach to linearization that results in an algorithm for computing δ from the appropriate fixed point function, and incidentally provides a clear demonstration that, in the conventional view, the linearized T has only a single isolated eigenvalue $\delta > 1$.

2. THE DOUBLING TRANSFORMATION

Initially, let us consider functions as shown in Fig. 1 which are zero at the origin, have a single maximum, are continuous, monotone increasing below the maximum, and monotone decreasing above the maximum.

Since we are interested in asymptotically long sequences of iterates, not all of the function as shown in Fig. 1 is relevant. For example, only the first iterate can be greater than the maximum $\phi_{\max}$ of ϕ. Asymptotically then, we lose no information if we consider x to be bounded above by $\phi_{\max}$. Furthermore, if $x_0 < \phi(\phi_{\max})$, a sequence of monotone increasing iterates will occur until an iterate eventually falls in the range

$$\phi(\phi_{\max}) < x < \phi_{\max} \tag{2.1}$$

This defines the asymptotic range of iterates; once an iterate is inside this range all subsequent iterates are inside this range.

Asymptotically then, the only relevant part of ϕ is inside the square shown in Fig. 1 with x and $\phi(x)$ in the range (2.1).

A suitable change of variable and a rescaling (amounting to a conjugacy) may always be chosen so that we need only consider functions defined in the unit square $0 < x < 1$; $0 < \phi(x) < 1$. We also find it convenient to apply an additional conjugacy $g(x) = 1 - x$ [Eq. (1.7)], which has the effect (shown in Fig. 2) of turning the unit square upside down and back-to-front. We shall justify this convenience shortly.

Thus we are led to consider the class of maps

$$\begin{aligned} \mathcal{C} = \{\phi \mid \phi : [0,1] \to [0,1];\ &\phi(0) = 1; \text{ monotone decreasing} \\ &\text{to } \phi(b) = 0 \text{ on } 0 \leqslant b < 1 \text{ and monotone increasing} \\ &\text{to } \phi(1) = a < 1 \text{ on } b \leqslant x \leqslant 1\} \end{aligned} \tag{2.2}$$

Since we are interested in iterates obtained from functions in $\mathcal{C}$ it is natural to consider iterates of functions themselves in $\mathcal{C}$. For example, in Fig. 2 we have sketched the second iterate $\phi^{(2)}$ [Eq. (1.4)] of a typical $\phi \in \mathcal{C}$ for the particular case where

$$a = \phi(1) < x^* = \phi(x^*) \tag{2.3}$$

The interesting feature in this case, as can be seen from Fig. 2, is that iterates obtained from $\phi^{(2)}$ below (above) x^* always remain below (above) x^*. Thus the function ϕ is restricted in such a way that only even cycles can appear so that one is including here the region of bifurcating 2^n-cycles.

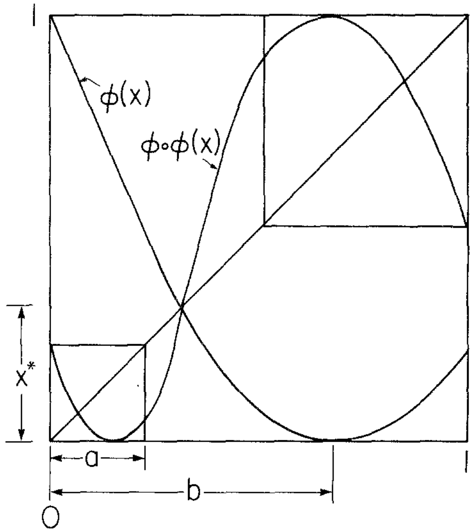

Fig. 2. Form of conjugated ϕ. The relevant parts of $\phi \circ \phi(x)$ are inside the two small squares and are conjugates of one another.

(Notice in particular that when $a = 0$ and $b = 1$, one has a stable two-cycle $0, 1$.) Also, as shown in Appendix A, the two disjoint pieces of $\phi^{(2)}$ (above and below x^*) are conjugate to one another so that one need only focus attention on one piece. Furthermore, for reasons given above, only one of the two conjugate pieces of $\phi^{(2)}$ shown in the little boxed squares is asymptotically relevant. It will be noticed in particular that the piece of $\phi^{(2)}$ on $0 < x < a$ resembles a scaled-down version of a function in the class $\mathcal{C}$. By an obvious change of scale we are then led naturally to consider the "doubling transformation" $T: \mathcal{C} \to \mathcal{C}$ defined by

$$T\phi(x) = a^{-1}\phi(\phi(ax)), \qquad 0 < a = \phi(1) < 1 \tag{2.4}$$

In view of the above remarks, all questions with respect to the behavior of long sequences of iterates (1.1) can equally well be answered by studying the iterated functions

$$\phi_k \equiv T^k\phi, \qquad k = 1, 2, \ldots \tag{2.5}$$

obtained by repeated application of (2.4), but only so long as our conjugacy requirement

$$a_k = \phi_k(1) < x_k^* = \phi_k(x_k^*) \tag{2.6}$$

is satisfied.

When condition (2.6) is violated it makes no sense to continue application of the doubling transformation T. The question then arises as to the circumstances under which the process (2.5) can be continued indefinitely. This question is discussed in the following section.

We note in passing that (2.4) is similar to the universal transformation of Feigenbaum[3] and also to the doubling transformation of Collet, Eckmann, and Lanford.[4] These authors choose to consider symmetric functions on $[-1, 1]$ with a single maximum at the origin. These functions without the symmetry requirement, are in fact (as shown in Appendix B) conjugate to the upper relevant portion of $\phi^{(2)}$ shown in Fig. 2.

The advantage of (2.4) is that all quantities are nonnegative and as a result there are no ambiguities in sign when discussing power law conjugacies. These conjugacies, as we will see, play a particularly important role in what follows.

3. CONVERGENCE TO THE FIXED POINT OF T AND POWER LAW CONJUGACIES

Suppose for the sake of argument that we have a one-parameter family of maps $\phi^{\lambda} \in \mathcal{C}$ such that for an increasing sequence of values $\bar{\lambda}_n$, $n = 1, 2, \ldots$, $\phi^{\bar{\lambda}_n}$ has a (super-) stable 2^n-cycle that includes the minimum. We will assume in addition that $\bar{\lambda}_n \uparrow \lambda_\infty < \infty$ as $n \to \infty$ and that $\lambda = \bar{\lambda}_1$ corresponds to $a = 0$, $b = 1$ in $\mathcal{C}$.

Now in general, it is easily seen from (2.4) that if ϕ has a stable $2p$ cycle, $T\phi$ has a stable p cycle. It follows that $T^n\phi^{\bar{\lambda}_n}$ has a superstable two-cycle and hence that

$$\phi_n^{\bar{\lambda}_n}(1) = T^n\phi^{\bar{\lambda}_n}(1) = 0 \tag{3.1}$$

Similarly if $\phi^{\lambda'}$ has an even non-2^n cycle, say a $q2^n$ cycle where q is odd, $T^n\phi^{\lambda'}$ has a q cycle which is odd and hence from the remarks of the previous section the condition (2.6) is violated after n applications of T. In this case, $\lambda' > \lambda_\infty$, and as remarked previously, it makes no sense to continue the doubling transformation. Similarly, from (3.1) and (2.4) $T^{n+1}\phi^{\bar{\lambda}_n}$ is not defined so the repetition of the doubling transformation must also cease, at least for values $\bar{\lambda}_n < \lambda_\infty$ corresponding to superstable 2^n-cycles.

Assuming there are no pathologies it seems reasonable to assert on the basis of "continuity" that the doubling transformation cannot be applied indefinitely when $\lambda \neq \lambda_\infty$. Although there is, to the authors' knowledge, no proof of this assertion, it has been adequately demonstrated numerically,[3] and moreover, that when $\lambda = \lambda_\infty$,

$$T^n\phi^{\lambda_\infty} \to f \qquad \text{as } n \to \infty \tag{3.2}$$

which is a fixed point of T. That is, from (2.4)

$$f(x) = a^{-1}f(f(ax)), \qquad a = f(1) \tag{3.3}$$

The existence of fixed point functions (3.3) has been proved in certain cases[4,5] and it is commonly believed that the particular fixed point approached in the limit (3.2) depends solely on the shape of ϕ^{λ_∞} at its minimum. We shall see in a moment that this is not strictly true but nevertheless from a certain viewpoint, the claimed "universality" is somewhat wider than previously thought.

In a search for universal behavior it seems natural to begin with the notion of conjugacy. Thus consider a sequence of functions

$$\{\phi_k = T^k\phi, k = 1,2, \ldots\} \tag{3.4}$$

obtained by applying T to a particular initial function ϕ.

Now make a proper conjugacy [by this we shall mean $g(0) = 0$ and $g(1) = 1$ in (1.7)] of the initial ϕ:

$$\hat{\phi} = g \circ \phi \circ g^{-1} \tag{3.5}$$

and apply T to $\hat{\phi}$ to obtain the sequence

$$\{\hat{\phi}_k = T^k\hat{\phi}, k = 1,2 \ldots\} \tag{3.6}$$

Note that if

$$\hat{\phi}_k = g_k \circ \phi_k \circ g_k^{-1} \tag{3.7}$$

where g_k is a proper conjugacy, then

$$\hat{\phi}_k(1) = g_k \circ \phi_k(1) \tag{3.8}$$

and after some algebra,

$$\hat{\phi}_{k+1} = T\hat{\phi}_k = g_{k+1} \circ \phi_{k+1} \circ g_{k+1}^{-1} \tag{3.9}$$

where

$$g_{k+1}(x) = [g_k \circ \phi_k(1)]^{-1} g_k(\phi_k(1)x) \tag{3.10}$$

and

$$g_{k+1}^{-1}(x) = [\phi_k(1)]^{-1} g_k^{-1}([g_k \circ \phi_k(1)]x) \tag{3.11}$$

Since the initial functions ϕ and $\hat{\phi}$ are related by a conjugacy (3.5) it then follows by induction from (3.7) and (3.9) that ϕ_k and $\hat{\phi}_k$ are conjugates (3.7) of one another, where from iteration of (3.10)

$$g_k(x) = [g(P_k)]^{-1} g(P_k x) \tag{3.12}$$

and

$$P_k = \prod_{l=0}^{k-1} \phi_l(1) \qquad [\phi_0 \equiv \phi] \tag{3.13}$$

In order that the sequence $\{\phi_k\}$ not terminate it is necessary, from (2.4) and (2.6), that $0 < \phi_l(1) < 1$, $l = 0, 1, 2 \ldots$. The P_k, therefore, become arbitrarily small and from (3.12) it follows that g_k depends solely on the behavior of $g(x)$ near $x = 0$.

In particular, if ϕ and $\hat{\phi}$ depend upon x in the same way at $x = 0$ and at their respective minima, then $g(x) \sim x$ as $x \to 0$. If this is the case, then from (3.12) and (3.7)

$$g_k(x) \sim x \qquad (\forall x) \quad \text{and} \quad \phi_k \sim \hat{\phi}_k \qquad \text{as } k \to \infty \tag{3.14}$$

If on the other hand g were chosen so that it approached zero like x^p for some power p as x approached zero, then for all $0 \leqslant x \leqslant 1$,

$$g_k(x) \sim x^p \quad \text{and} \quad \hat{\phi}_k(x) \sim \left[\phi_k(x^{1/p})\right]^p \qquad \text{as } k \to \infty \tag{3.15}$$

That is, $\hat{\phi}_k$ and ϕ_k in this case are asymptotically related by a power law conjugacy.

As a particular case, if the initial function ϕ were chosen to be a fixed point of T, then the above argument shows that any conjugacy of the fixed point function converges under repetition of T to a power law conjugacy of that fixed point function. It is also easily verified that a power law conjugacy

$$\hat{f}(x) = \left[f(x^{1/p})\right]^p \tag{3.16}$$

of a fixed point f of T with reduction parameter $a = f(1)$, is also a fixed point of T with reduction parameter a^p.

In summary then, we have shown that conjugacy classes, of functions that converge to fixed points of T, converge under successive application of T to conjugacy classes of the respective fixed points of T, and that asymptotically, the only relevant conjugacies are power law conjugacies (3.16).

We shall demonstrate in the following sections that δ [Eq. (1.6)] may be calculated directly from a fixed point function and that power law conjugacy classes are universal in the sense that functions related by a transformation (3.16) have the same value of δ [Eq. (1.6)]. It then follows from the above that "δ" depends not only on how a function behaves near its (zero) minimum but also on how it behaves near its maximum (at $x = 0$). This dependence, however, is only on the product of the two exponents characterizing the behavior of the function at its two extrema.

For example, in the above notation, the function

$$\phi(x) = \left(1 - \lambda_{\infty} x^2\right)^2 \qquad (0 < \lambda < 1) \tag{3.17}$$

has a quadratic minimum and converges to a fixed point of T but does not have Feigenbaum's $\delta = 4.6692016\ldots$. The value of $\delta = 8.3\ldots$ in this case is characteristic of "a quartic" and this follows from the fact that $\phi(x)$ (3.17) is a power law conjugate of a "legitimate" quartic. That is,

$$\left[\phi(x^{1/2})\right]^2 = (1 - \lambda_{\infty} x)^4 \tag{3.18}$$

In general, functions of the form

$$\phi(x) = (1 - \lambda_{\infty} x^s)^t \qquad \text{with } st = p \text{ fixed} \tag{3.19}$$

form a universality class in the sense that they all converge to a conjugacy class of fixed points of T characterized by a "δ" that depends only on p.

4. LINEARIZATION AND LOCAL DIVERGENCE FROM A FIXED POINT FUNCTION

In the previous section we have seen that fixed points (3.3) of the doubling transformation (2.4) are unstable in the sense that parametrized functions ϕ^{λ} do not iterate to fixed points of T unless λ is chosen precisely to be the accumulation point λ_{∞} of bifurcating 2^n-cycles (1.6). To investigate the nature of the instability it is then natural to consider the action of T on some function "close to" a fixed point function. More precisely, for small ϵ and f a fixed point (3.3) of T we obtain from (2.4) the linearized form of the doubling transformation

$$T \circ (f + \epsilon h)(x) = f(x) + \epsilon \pounds_f \circ h(x) + O(\epsilon^2) \tag{4.1}$$

where the linear operator $\pounds_f$ is defined by

$$\pounds_f \circ h(x) = a^{-1}\{h(1)(xf'(x) - f(x)) \\ + f'(f(ax))h(ax) + h(f(ax))\} \tag{4.2}$$

In order to preserve the normalization $f(0) = 1$ we require

$$h(0) = 0 \tag{4.3}$$

Assuming that f is regular at the origin [or at least that $xf'(x) \to 0$ as $x \to 0+$] the first term in (4.2) is essentially just a rescaling to give $\pounds_f \circ h(0) = 0$.

Since $\pounds$ is linear, repeated application of (4.1) gives

$$T^n \circ (f + \epsilon h)(x) = f(x) + \epsilon \pounds_f^n \circ h(x) + O(\epsilon^2) \tag{4.4}$$

and if $\pounds$ has a single isolated maximum eigenvalue $\delta > 1$, it follows that

$$T^n \circ (f + \epsilon h)(x) \sim f(x) + \epsilon\delta^n H(x) + O(\epsilon^2) \tag{4.5}$$

for some function $H(x)$ depending on $h(x)$ and the principal eigenfunction of $\pounds$. In other words, the rate of divergence from a fixed point function f is governed by the maximum eigenvalue of the operator $\pounds_f$ obtained from T by linearizing around this fixed point function.

The existence of such an eigenvalue ($\delta > 1$) has been proved in the special cases mentioned previously.[4] In the following section we give a formula for actually computing δ from its associated fixed point function f.

On the subject of universality it turns out that power law conjugacies also play an important role in the sense that if f and $\hat{f}$ are related by a power law conjugacy (3.16), the associated linear operators $\pounds_f$ and $\pounds_{\hat{f}}$ have the same point spectrum, and in particular the same maximum eigenvalue δ (assuming that it exists).

To prove this we simply note that for the operator S defined by

$$S \circ \psi(x) = \left[f(x^{1/p}) \right]^{p-1} \psi(x^{1/p}) \tag{4.6}$$

we have from (4.2) the operator identity

$$\pounds_{\hat{f}} S = S \pounds_f \tag{4.7}$$

It follows that if ψ is an eigenfunction of $\pounds_f$ with eigenvalue λ then $S\psi$ is an eigenfunction of $\pounds_{\hat{f}}$ also with eigenvalue λ. Reversing the roles of $\hat{f}$ and f then shows that $\pounds_f$ and $\pounds_{\hat{f}}$ have identical point spectra.

Power law conjugacy classes, therefore, have the same δ so in some sense these classes form universality classes. From the above remarks and those of the previous section it may well be that these classes are the only universality classes.

There are several ways to show that the "divergence parameter" δ defined above also gives the rate of convergence (1.6) of the λ_n parameters to λ_∞. Perhaps the simplest way is to consider the one-parameter family of maps ϕ^λ defined by

$$\phi^\lambda(x) = f(\lambda x) \tag{4.8}$$

where f is a fixed point function (3.3) of T.

In this case $\lambda_\infty = 1$, and modulo some regularity assumption, there will be a monotonic increasing sequence of values $\bar{\lambda}_n < 1$, $n = 1, 2 \ldots$ converging to unity and corresponding to superstable 2^n-cycles of ϕ^λ.

Setting $\lambda = \bar{\lambda}_n$ in (4.8) we obtain

$$\begin{aligned} \phi^{\bar{\lambda}_n}(x) &= f\left(x + (\bar{\lambda}_n - 1)x\right) \\ &= f(x) + (\bar{\lambda}_n - 1)xf'(x) + O(\bar{\lambda}_n - 1)^2 \qquad \text{as } n \to \infty \end{aligned} \tag{4.9}$$

Substitution into (4.5) then gives [with $h(x) = xf'(x)$]

$$T^n \circ \phi^{\bar{\lambda}_n}(x) \sim f(x) + (\lambda_n - 1)\delta^n H(x) + O(\lambda_n - 1)^2 \tag{4.10}$$

Setting x equal to unity and noting (3.1) we then find that

$$1 - \lambda_n \sim C\delta^{-n} \qquad \text{as } n \to \infty \tag{4.11}$$

where $C = f(1)/H(1)$.

5. COMPUTATION OF THE DIVERGENCE PARAMETER δ

In order to compute δ we begin with (4.2) expressed in the form

$$£_f \circ h_0(x) = a^{-1}h_0(1)\psi(x) + h_1(x) \tag{5.1}$$

where

$$\psi(x) = xf'(x) - f(x) \tag{5.2}$$

and

$$h_1(x) = a^{-1}\left[h_0(f(ax)) + h_0(ax)f'(f(ax)) \right] \equiv A \circ h_0(x) \tag{5.3}$$

Differentiating the fixed point equation (3.3) with respect to x we obtain

$$f'(x) = f'(f(ax))f'(ax) \tag{5.4}$$

and assuming that f is a regular fixed point function [by which we mean $f'(0) \neq 0$ and finite] we have that $f'(1) = 1$ and from (5.1) and (5.2),

$$£_f \circ \psi(x) = a^{-1}\psi(x) \tag{5.5}$$

Repeated application of (5.1) then gives

$$£_f^l \circ h_0(x) = \alpha_l \psi(x) + h_l(x) \tag{5.6}$$

where

$$h_l(x) = A^l \circ h_0(x), \qquad l = 1, 2, \ldots \tag{5.7}$$

and from the normalization condition $£_f^l \circ h_0(0) = 0$ we have [since $\psi(0) = -1$]

$$\alpha_l = h_l(0) \tag{5.8}$$

Setting

$$h_l(x) = n_l(x)f'(x), \qquad l = 0, 1, 2 \ldots \tag{5.9}$$

in (5.7) and using (5.4) we obtain, after iterating (5.7), the expression

$$n_l(x) = a^{-l} \sum_{k=0}^{2^l - 1} \frac{n_0\left(f^{(k)}(a^l x)\right)}{f^{(k)'}(a^l x)} \tag{5.10}$$

where $f^{(k)'}$ denotes the derivative of the kth iterate $f^{(k)}$ of f and $f^{(0)}(x) \equiv x$.

Now since $a < 1$ it follows that $n_l(x)$ is asymptotically independent of x (in the unit interval) as $l \to \infty$ so in the above expressions we can replace $n_l(x)$ by $n_l(0)$ for large l.

Combining (5.2), (5.3), (5.6), (5.8), and (5.9) we then have, as $l \to \infty$,

$$\pounds_f^l \circ n_0(x) f'(x) \sim n_l(0)\{f'(x) + f'(0)[xf'(x) - f(x)]\} \tag{5.11}$$

Since this result holds for all $n_0(x)$, it follows that

$$n_l(0) \sim A\delta^l \qquad \text{as } l \to \infty \tag{5.12}$$

where δ is the maximum eigenvalue of $\pounds_f$, or more precisely

$$\delta = \lim_{l \to \infty} [n_{l+1}(0)/n_l(0)] \tag{5.13}$$

At first sight it may appear from (5.10) that δ defined in this way depends on the function $n_0(x)$. In Appendix C, however, we show that this is not the case.

A convenient choice for $n_0(x)$ is $n_0(x) = x$, giving, from (5.10) and (5.13),

$$\delta = \lim_{l \to \infty} [X_{l+1}/X_l] \tag{5.14}$$

where

$$X_l = a^{-l} \sum_{k=1}^{2^l - 1} \frac{f^{(k)}(0)}{f^{(k)'}(0)} \tag{5.15}$$

Summing this series as a method for calculating the divergence parameter is quite efficient. Thus given an algorithm for computing $f(x)$ and making use of the fact that

$$f^{(k)'}(x) = \prod_{s=1}^{k-1} f'(f^{(s)}(x)) \tag{5.16}$$

we can write X_0 as a kind of continued fraction:

$$a^l X_l = \frac{1}{f'(0)} \left\{ f(0) + \frac{1}{f'(f(0))} \left[f(f(0)) \right.\right.$$
$$\left.\left. + \frac{1}{f'(f(f(0)))} [f(f(f(0))) + \cdots \right\} \tag{5.17}$$

where there are $2^l - 1$ terms in the sum.

For example, using Feigenbaum's algorithm for computing the regular fixed point function with a quadratic minimum and using the relation (B.5) we obtain from (5.17) the estimates

$$\begin{aligned} X_2/X_1 &= 4.671, \qquad X_3/X_2 = 4.66922 \\ X_4/X_3 &= 4.669204, \qquad X_5/X_4 = 4.6692019 \end{aligned} \tag{5.18}$$

compared to the exact value of $\delta = 4.6692016\ldots$.

Finally it should be remarked that the assumption of regularity for f ($f'(0)$ finite) is a convenience rather than a necessity. Thus since $n_l(x)$ above is asymptotically independent of x we can substitute say $x = 1$ if there is trouble at the origin. In any event if a fixed point function behaves as a power of x (other than x itself) at the origin, the function can be made regular by a power law conjugacy which at the same time preserves the value of δ.

6. DISCUSSION

In this paper we have presented an alternative view of universal properties of maps on an interval which highlights the role played by power law conjugacies.

To take account of such conjugacies we find it convenient to consider functions defined on the unit interval with a single zero minimum and a single maximum of unity at the origin as shown in Fig. 2. Such functions are conjugate to those considered by other authors as are fixed points of our respective doubling transformations.

We show that conjugacies of a fixed point function f of our doubling transformation T converge in general to power law conjugacies of f and moreover that local divergence from f is governed by a parameter δ that is invariant under power law conjugacies.

This analysis shows a departure from the conventional outlook on the properties of the divergence parameter. It is usually asserted that this parameter is a function only of the behavior of the iterating function near its interior extremum, in our case the zero minimum. However, from the aforementioned invariance, we show that the parameter also depends on the behavior of the iterating function near its maximum (at $x = 0$). This nevertheless leads to a widening of the concept of universality to power law conjugacy classes in which the divergence parameter is a function only of the product of the powers with which the interating function approaches unity at the maximum and zero at the minimum. If one standardizes either one of these powers to be unity, the divergence parameter is, of course, only a function of the other.

Finally, by choosing a fixed point function with standardized power unity at the maximum we construct a very effective method for computing the divergence parameter from iterates, and the derivative of iterates, of the fixed point function evaluated at the origin.

The analogy between the doubling transformation and the renormalization group transformation in critical phenomena is striking and has been discussed previously.(3)

In our approach one can think of initializing maps of the form, say

$$\phi^{\lambda}(x) = |1 - \lambda x^s|^t \tag{6.1}$$

as representing a class of "Hamiltonians" incorporating the "temperature" λ. Under repetition of the (renormalization) doubling transformation T, the Hamiltonian (6.1) converges to a fixed point Hamiltonian (of T) when λ is set equal to the "critical temperature" λ_{∞}. Local divergence from the fixed point Hamiltonian is governed by relevant eigenvalues $\delta > 1$ of the linearized transformation. δ determines critical exponents, and universality classes consist of those Hamiltonians which give rise to the same critical exponents.

In the simple case (6.1) we have shown that Hamiltonians with the product st fixed belong to the same universality class. More generally we have shown invariance of δ under conjugacy. It would be interesting if the notion of conjugacy and its relevance to universality could be adapted to real physical systems.

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APPENDIX A. CONJUGATE PARTS OF $\phi \circ \phi$

The rescaled part of $\phi \circ \phi$ on $0 < x < a = \phi(1)$ is given by

$$\phi_1(x) = a^{-1}\phi(\phi(ax)) \tag{A.1}$$

and on $c = 1 - \phi(\phi(1)) \leqslant x \leqslant 1$ by

$$\phi_2(x) = c^{-1}[1 - \phi(\phi(1 - cx))] \tag{A.2}$$

These correspond respectively to the lower and upper small squares in Fig. 2. Equation (A.1) is just the doubling transformation (2.4).

To show that ϕ_2 is conjugate to ϕ_1 define

$$g(x) = c^{-1}[1 - \phi(ax)] \tag{A.3}$$

which is easily seen to be monotone on [0, 1] and thus may be inverted.

Let

$$h(x) = a^{-1}\phi(1 - cx) \tag{A.4}$$

so that $h(0) = 1$ and $h(b/c) = 0$ corresponding to the zero minimum of ϕ.

Solving (A.3) for $\phi(ax)$ and substituting into (A.1) we obtain

$$\phi_1(x) = a^{-1}\phi(1 - cg(x)) = h(g(x)) \tag{A.5}$$

Similarly, from (A.2), (A.3), and (A.4) we have

$$\phi_2(x) = c^{-1}\left[1 - \phi(ah(x))\right] = g(h(x)) \tag{A.6}$$

Comparing (A.5) and (A.6) shows that

$$\phi_2 = g \circ \phi_1 \circ g^{-1} \tag{A.7}$$

and Bob's your uncle.

APPENDIX B. RELATION TO FEIGENBAUM'S UNIVERSAL FUNCTION

The "Universal Transformation" of Feigenbaum[3] and the "doubling transformation" of Collet, Eckmann and Lanford[4] are closely related to the second kind of functional transformation defined by (A.2). A fixed point function of this transformation would satisfy

$$\begin{aligned} f(x) &= \alpha\left(1 - f\left(f(1 - \alpha^{-1}x)\right)\right) \\ \alpha &= \left[1 - f(f(1))\right]^{-1} \end{aligned} \tag{B.1}$$

Let $x = b$ specify the minimum of $f(x)$ so that $f(b) = 0$, and note, therefore, from (B.1) that

$$f(1 - b/\alpha) = 0$$

and thus that

$$b = \alpha(1 + \alpha)^{-1}$$

Using these results, the conjugacy

$$g(x) = x - b, \qquad g^{-1}(x) = x + b$$

transforms (B.1) into

$$\hat{f}(x) = -\alpha\hat{f}\left(\hat{f}(-\alpha^{-1}x)\right) \tag{B.2}$$

where

$$\hat{f}(x) = g\left(f\left(g^{-1}(x)\right)\right) = f(x + b) - b$$

This, with the internally consistent assumption that $\hat{f}$ is even in x, yields the functional equation of the above-mentioned authors:

$$\hat{f}(x) = -\alpha\hat{f}\left(\hat{f}(\alpha^{-1}x)\right) \tag{B.3}$$

A solution to this equation, the fixed point equation of the doubling transformation, will thus be conjugate to a solution of (3.3) since both are conjugate to a solution of (B.1). Since the form of the fixed point equation for the doubling transformation (B.3) and the equation (3.3) are the same, the conjugacy connecting them must be a power law conjugacy according to the results of Section 3.

To find the power of the conjugacy notice that the reduction parameter α of (B.2) is the same as the reduction parameter of (B.1). Thus we seek a connection between the reduction parameter of (B.1) and the reduction parameter of (3.3).

If we iterate (B.1) 2^N times we obtain

$$f^{2^N}(x) = \alpha\left[1 - f^{2^N}\left(f^{2^N}(1-\alpha^{-1}x)\right)\right]$$

Let $x = b$ and subtract b from both sides. Recalling that $\alpha = b/(1-b)$ we obtain

$$\alpha^{-1} = -\left[b - f^{2^N}\left(f^{2^N}(b)\right)\right]\left[b - f^{2^{N-1}}\left(f^{2^{N-1}}(b)\right)\right]^{-1}$$

That is,

$$\alpha^{-1} = -\left[b - f^{2^{N+1}-1}(0)\right]\left[b - f^{2^N-1}(0)\right]^{-1} \tag{B.4}$$

for any N.

Again, the results of Section 3 show that repeated application of the functional transformation (2.4) to the function f will asymptotically approach a solution of (3.3). Thus for large N

$$f^{2^N}(0) = f\left(f^{2^N-1}(0)\right) = a^N \tag{B.5}$$

The quantity a^N is arbitrarily small for large N, which implies that $f^{2^N-1}(0)$ must then be nearly equal to b where f has its single zero minimum. We must provide the additional information of the form of f at its minimum to complete the connection of the reduction parameters through (B.4).

Let us presume the typical case where the minimum is quadratic. Expanding (B.5) about the minimum of f we obtain

$$f''(b)\left[b - f^{2^N-1}(0)\right]^2 \sim a^N$$

Substitution in (B.4) yields

$$\alpha = -a^{-1/2}$$

and therefore a solution to the fixed point of our functional equation (3.3) can be obtained from Feigenbaum's solution to (B.2) by the power law conjugacy

$$f(x) = \left[\hat{f}\,(x^{1/2})\right]^2$$

where the reduction parameter is now

$$a = \alpha^{-2} = 0.159628442\ldots$$

Feigenbaum[3] provides a set of coefficients for a power series expansion of $\hat{f}$. We repeat those coefficients here to correct a previous typographical error. To compute f rapidly make the above power law conjugacy of Feigenbaum's

$$\hat{f}\,(x) = 1 + \sum_{i=1}^{7} g_i x^{2i}$$

That is, let

$$f(x) = \left(1 + \sum_{i=1}^{7} g_i x^{i}\right)^2 \tag{B.6}$$

where

$$\begin{aligned}
g_1 &= -1.527632997\\
g_2 &= 1.048151943 \times 10^{-1}\\
g_3 &= 2.670567349 \times 10^{-2}\\
g_4 &= -3.527413864 \times 10^{-3}\\
g_5 &= 8.158191343 \times 10^{-5}\\
g_6 &= 2.536842339 \times 10^{-5}\\
g_7 &= 2.687772769 \times 10^{-6}
\end{aligned}$$

Equation (B.6) was used to obtain the estimates (5.18) for the divergence parameter.

APPENDIX C. ASYMPTOTIC INDEPENDENCE OF $n_{l+1}(x)/n_l(x)$ ON $n_0(x)$

We begin by substituting a^m for x in (5.10) and rewriting the sum over k as

$$\begin{aligned}
n_l(a^m) &= a^{-l} \sum_{p=0}^{2^{l-m}-1} \sum_{k=p^{2^m}}^{(p+1)2^m-1} \frac{n_0\left(f^{(k)}(a^{l+m})\right)}{f^{(k)\prime}(a^{l+m})}\\
&= a^{-l} \sum_{p=0}^{2^{l-m}-1} \sum_{t=0}^{2^m-1} \frac{n_0\left(f^{(p2^m+t)}(a^{l+m})\right)}{f^{(p2^m+t)\prime}(a^{l+m})}
\end{aligned} \tag{C.1}$$

Two relations which follow from the fixed point functional equation (3.3) are, for p and t integers:

$$f^{(p2^m+t)}(a^{l+m}) = f^{(t)}\left(a^m f^{(p)}(a^l)\right)$$

and

$$f^{(p2^m+t)'}(a^{l+m}) = \left[f^{(t)'}\left(a^m f^{(p)}(a^l)\right)\right]\left[f^{(p)'}(a^l)\right]$$

Making use of these relations we then have

$$n_l(a^m) = a^{-l}\sum_{p=0}^{2^{l-m}-1}\left[f^{(p)'}(a^l)\right]^{-1}\sum_{t=0}^{2^m-1}\frac{n_0\left[f^{(t)}\left(a^m f^{(p)}(a^l)\right)\right]}{f^{(t)'}\left(a^m f^{(p)}(a^l)\right)}$$

$$= a^{-(l-m)}\sum_{p=0}^{2^{l-m}-1}\left[f^{(p)'}(a^l)\right]^{-1} n_m\left(f^{(p)}(a^l)\right) \tag{C.2}$$

where in the last step we have used the definition (5.10) of $n_m(x)$.

Now from (5.10) it follows that for large l $n_l(x) \sim n_l(0)$ for all x fixed in the unit interval. It then follows from (C.2) that for $l \gg m \gg 1$,

$$n_l(0) \sim \Lambda_{l-m} n_m(0) \tag{C.3}$$

where

$$\Lambda_s = a^{-s}\sum_{p=0}^{2^s-1}\left[f^{(p)'}(0)\right]^{-1} \tag{C.4}$$

It will be noted from (5.10) that Λ_s is precisely $n_s(0)$ for the particular choice $n_0(x) = 1$ and that from (5.12) and (C.3)

$$\delta = \lim_{s\to\infty} \Lambda_{s+1}/\Lambda_s$$

From (C.3), however, it also follows that any choice of $n_0(x)$ in (5.10) may be used to evaluate δ.

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